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PROCEEDINGS

OF

THE ROYAL IRISH ACADEMY.

1844–45. No. 50.

January 27, 1845.

SIR WM. R. HAMILTON, LL.D., President, in the Chair.

The Correspondence (see *Appendix*, No. I.) was read from the Minutes of Council.

The Rev. Charles Graves read the continuation of his paper on Algebraic Triplets.

The triplet $x + \iota y + \iota^2 z$, or (x, y, z), being employed to represent the right line drawn from the origin to the point whose rectangular coordinates in space are x, y, and z, it becomes a matter of interest to determine the position of the right line which represents the product of two such triplets. For this purpose it will be convenient to lay down some definitions.

1. The *symmetric axis* is the right line drawn from the origin, so as to make equal angles with the three positive portions of the axes of coordinates.

2. The *symmetric plane* is a plane passing through the origin, and perpendicular to the symmetric axis.

3. The *modular plane* is the plane containing the axis of x and the symmetric axis.

4. The *radius*, r, of the triplet (x, y, z) is the right line drawn from the origin to the point x, y, z.

5. Its *vector angle*, ρ, is the angle between the radius and the symmetric axis.

6. The *amplitude*, ω, of the triplet (x, y, z) is the angle between the modular plane and the plane containing the radius and the symmetric axis.

We are able to determine the radius, vector angle, and amplitude, of the triplet (r_2, ρ_2, ω_2), which is formed by multiplying together two triplets (r, ρ, ω) and (r_1, ρ_1, ω_1), by means of the following equations, in which s stands for the angle between the symmetric axis and the positive portion of any one of the axes of coordinates.

$$rr_1 \cos\rho \cos\rho_1 = r_2 \cos s \cos\rho_2 \qquad (1)$$

$$rr_1 \sin\rho \sin\rho_1 = r_2 \sin s \sin\rho_2 \qquad (2)$$

$$\omega + \omega_1 = \omega_2. \qquad (3)$$

It appears from the first two of these equations that whether we call $\frac{r\cos\rho}{\cos s}$, or $\frac{r\sin\rho}{\sin s}$, the modulus of the triplet, it will be true to say that *the modulus of the product is equal to the product of the moduli of the factors.*

The third equation asserts, that *the amplitude of the product is equal to the sum of the amplitudes of the factors.*

As the real unit is supposed to be placed on the axis of x, we shall obtain the following theorem by dividing the second of the preceding equations by the first:

The tangents of the vector angles of the real unit, of the two factors, and of the product, form a proportion.

Mr. Graves stated that Sir Wm. Hamilton had been the first to announce that if the real unit line, the factors, and the product line, be projected upon the symmetric axis, the projections will form a proportion in the simple sense of that term: whilst the projections of the same lines on the symmetric plane form a proportion, according to the higher sense in which Mr. Warren uses the same word.

The former of these theorems is merely the geometric interpretation of equation (1): the latter of equations (2) and (3).

The following cases deserve special attention:

If either of the factor lines coincides with the symmetric axis, the product line must also coincide with it.

If either of the factor lines is contained in the symmetric plane, the product line must also be contained in it.

But if one coincides with the symmetric axis, and the other with the symmetric plane, the product line will vanish.

Startling as these consequences may appear, they are to be explained by reference to the geometric meaning of the symbols ι and ι^2; both of which, according to the interpretation assigned to them, are inoperative to move a right line out of either the symmetric plane or the symmetric axis.

The analytical difficulty raised by the last case seems to force us to admit that the vanishing of a product does not necessarily imply the vanishing of a factor. In the present instance it is caused by the vanishing of one of the moduli of multiplication belonging to each of the factors.

The length of the product line is equal to the product of the lengths of the factor lines only in the case where the vector angle of one of the factors is equal to the vector angle of the real linear unit (where ρ or $\rho_1 = s$).

Having thus interpreted the results of multiplication by means of the existing trigonometry, Mr. Graves proceeded to show how the use of a new kind of trigonometry gives increased symmetry and flexibility to the present theory of algebraic triplets.

The foundations of this new calculus are thus laid. Using the exponential development we shall find

$$e^{\iota\phi} = \lambda + \iota\mu + \iota^2\nu$$

where

$$\lambda = 1 + \frac{\phi^3}{1.2.3} + \frac{\phi^6}{1.2.3.4.5.6} + \&c.$$

$$\mu = \phi + \frac{\phi^4}{1.2.3.4} + \frac{\phi^7}{1.2.3.4.5.6.7} + \&c.$$

$$\nu = \frac{\phi^2}{1.2} + \frac{\phi^5}{1.2.3.4.5} + \frac{\phi^8}{1.2.3.4.5.6.7.8} + \&c.$$

Again,

$$e^{\iota^2\chi} = \lambda_{,} + \iota^2\mu_{,} + \iota\nu_{,}$$

where $\lambda_{,}$, $\mu_{,}$, $\nu_{,}$ are the same functions of χ that λ, μ, ν are of ϕ. Hence we have

$$e^{\iota\phi + \iota^2\chi} = \mathrm{A} + \iota\mathrm{M} + \iota^2\mathrm{N},$$

A standing for $\lambda\lambda_{,} + \mu\mu_{,} + \nu\nu_{,}$, M for $\lambda\nu_{,} + \mu\lambda_{,} + \nu\mu_{,}$, and N for $\lambda\mu_{,} + \mu\nu_{,} + \nu\lambda_{,}$.

The three functions, A, M, and N depending, each of them, on two variables, ϕ and χ, hold the same place in the present calculus that *cosine* and *sine* hold in the received trigonometry. And as the sum of the squares of *cosine* and *sine* is always equal to unity, so the equation

$$\mathrm{A}^3 + \mathrm{M}^3 + \mathrm{N}^3 - 3\,\mathrm{AMN} = 1$$

holds good, no matter what be the amplitudes ϕ and χ.

The importance of these formulæ in our theory of triplets is most obvious. For a triplet $x + \iota y + \iota^2 z$ may in general be thrown into the form $m(\mathrm{A} + \iota\mathrm{M} + \iota^2\mathrm{N})$, which, as we have seen, is equivalent to $me^{\iota\phi + \iota^2\chi}$. So that if m be called the modulus, and ϕ and χ the amplitudes, of the triplet, we shall find, on multiplying two triplets together, the following theorems to be true:

The modulus of the product is equal to the product of the moduli of the factors.

Either of the two amplitudes of the product is equal to the sum of the two corresponding amplitudes of the factors.

The modulus m is connected with the constituents of the triplet (x, y, z) by the following equation,

$$m^3 = x^3 + y^3 + z^3 - 3xyz,$$

with respect to which it is to be observed that the right hand member is the product of

$$x+y+z \text{ and } x^2+y^2+z^2-xy-yz-zx,$$
the two real moduli which have previously been shewn to belong to the triplet $(x, y, z.)$—See page 53.

Mr. Graves stated that he had obtained a multitude of formulæ concerning the functions A, M, and N, analogous to the fundamental formulæ of trigonometry. Amongst the more remarkable of these he pointed attention to one corresponding to the well-known theorem of Moivre.

By pursuing a similar course we may frame a theory of multiplets, admitting a like interpretation. In order to accomplish this we must assume a symbol κ, such that $\kappa^n(1)=1$, whilst $1, \kappa(1)$, $\kappa^2(1)$, $\kappa^3(1) \ldots$ &c. are looked upon as units absolutely differing in kind as much as unity differs from $\sqrt{-1}$. The development of $e^{\kappa\phi}$ into the form $a + \kappa\beta + \kappa^2\gamma +$ &c. will give us a set of n functions, a, β, $\gamma \ldots.$ each depending upon one variable ϕ: and again, the expansion of $e^{\kappa\phi+\kappa^2\chi+\kappa^3\psi+\ldots.}$ furnishes us with a series of n functions, A, B, Γ, &c, each depending upon $(n-1)$ variables ϕ, χ, ψ, &c. The multiplet $a+\kappa b+\kappa^2 c+$ &c. being now written in the form $m(\mathrm{A}+\kappa\mathrm{B}+\kappa^2\Gamma+$ &c.), which is equivalent to $me^{\kappa\phi+\kappa^2\chi+\kappa^3\psi+\&c.\ldots}$, it is evident that, if we call m the modulus, and ϕ, χ, ψ, &c. the amplitudes of the multiplet, we shall have the same theorems concerning moduli and amplitudes that have been already established in the case of the multiplication of couplets and triplets.

If, for instance, we form a quadruplet (w, x, y, z) by the aid of the symbol κ, which is a pure imaginary fourth root of positive unity, we shall find that the quantities $[w+x+y+z]$, $[(w+y)-(x+z)]$, and $[(w-y)^2+(x-z)^2]$ are all moduli of multiplication. The product of the three is equal to m^4.

Mr. Graves mentioned that his elder brother, Mr. John T. Graves, had been the first to conceive the notion of employing the functions λ, μ, and ν in the interpretation of this theory of triplets; but as they involve only one variable it is not possible to bring a triplet in general into the form

$$m(\lambda+\iota\mu+\iota^2\nu).$$

The President stated that the remarkable researches respecting algebraic triplets, made lately by Professor De Morgan and John T. Graves, Esq. in England, and here by the Rev. Charles Graves, had led him to perceive the following theorem :

If the three symbols ξ, η, ζ, or rather their squares and products, be supposed to satisfy the three following "equations of signification :"

$$\begin{aligned} \xi\,(b\eta + c\zeta) &= a\,(\eta^2 + \zeta^2), \\ \eta\,(c\zeta + a\xi) &= b\,(\zeta^2 + \xi^2), \\ \zeta\,(a\xi + b\eta) &= c\,(\xi^2 + \eta^2)\,; \end{aligned}$$

and if, by the help of these three equations, we eliminate any three of the six quadratic combinations ξ^2, η^2, ζ^2, $\xi\eta$, $\eta\zeta$, $\zeta\xi$, from the development of the "formula of multiplication,"

$$\begin{aligned} &(u\xi + v\eta + w\zeta)\,(x''\xi + y''\eta + z''\zeta) \\ &= (x\xi + y\eta + z\zeta)\,(x'\xi + y'\eta + z'\zeta), \end{aligned}$$

and then treat the three remaining combinations of the same set (ξ^2, &c.) as three entirely arbitrary and independent multipliers: the three separate equations thus obtained between the fifteen real quantities $a\ b\ c\ u\ v\ w\ x\ y\ z\ x'\ y'\ z'\ x''\ y''\ z''$ will be such, that whether we project the four lines (u, v, w), (x, y, z), (x', y', z'), (x'', y'', z''), on the axis $(a, b, c,)$ itself, or on the plane perpendicular to that axis, *the four projections thus obtained will in each case form a proportion ;* the proportionality of the projections on the axis being of the kind considered in ordinary algebra, and the proportionality of the projections on the plane perpendicular to the axis being of the kind considered by Mr. Warren ; that is to say, the lengths of these last projections are proportionals in the usual sense, and the rotation from the first to the second is equal to the rotation from the third to the fourth.

Sir W. Hamilton has been able to prove this theorem by treating the three real equations between the fifteen rectangular co-ordinates a, b, c, &c. according to the known methods of algebraical geometry. He has also arrived at simplifications of this proof by introducing the ordinary imaginary $\sqrt{-1}$, treated by the ordinary rules; but his first investigation, and the one which he prefers, has been founded on the rules of the calculus of quaternions, and consists in resolving by those rules the system of the two equations:

$$\frac{\alpha\rho'' + \rho''\alpha}{\alpha\rho' + \rho'\alpha} = \frac{\alpha\rho + \rho\alpha}{\alpha\upsilon + \upsilon\alpha}; \quad \frac{\alpha\rho'' - \rho''\alpha}{\alpha\rho' - \rho'\alpha} = \frac{\alpha\rho - \rho\alpha}{\alpha\upsilon - \upsilon\alpha};$$

in which

$$\begin{aligned} \alpha &= ia + jb + kc, \\ \upsilon &= iu + jv + kw, \\ \rho &= ix + jy + kz, \\ \rho' &= ix' + jy' + kz', \\ \rho'' &= ix'' + jy'' + kz'', \end{aligned}$$

$i, j, k,$ being (as in former communications respecting quaternions) three imaginary units connected by the nine non-linear relations

$$i^2 = j^2 = k^2 = -1; \; ij = k, jk = i, ki = j; \; ji = -k, kj = -i, ik = -j.$$

The phrase " equations of signification" is borrowed from Mr. De Morgan. If the theorem be particularized, so as to correspond to that gentleman's system of triplets, by making

$$u = 1, \quad v = w = 0, \quad a = -b = -c,$$

then the equations of signification reduce themselves to

$$\xi^2 = \eta\zeta, \quad \eta^2 = -\zeta\xi, \quad \zeta^2 = -\xi\eta,$$

and the formula of multiplication resolves itself into the three relations

$$\begin{aligned} x'' &= xx' + yz' + zy', \\ y'' &= xy' + yx' - zz', \\ z'' &= xz' + zx' - yy'. \end{aligned}$$

On the other hand, some of the results of the systems of the two Messrs. Graves may be reproduced by taking the same *unit-line*, $u = 1$, $v = w = 0$, but employing that other *axis* for which $a = b = c$. The equations of signification give then, more simply,

$$\xi^2 = \eta\zeta, \quad \eta^2 = \zeta\xi, \quad \zeta^2 = \xi\eta;$$

and the terms in x'', y'', z'', are all to be taken with positive signs. Sir William Hamilton pretends to no farther merit in the matter than to that of having sought to illustrate, by generalizing in one direction, the foregoing points of the theories of his friends.

February 10, 1845.

SIR Wm. R. HAMILTON, LL.D., President, in the Chair.

Robert Forster, Esq., William Le Fanu, Esq., Reverend George Longfield, John M. Neligan, M. D., William Justin O'Driscoll, Esq., Nicholas P. O'Gorman, Esq., Algernon T. Preston, Esq., and James Emerson Tennant, Esq., M. P., were elected Members of the Academy.

The President read a paper on Quaternions.—See *Appendix*, No. III.

A stone celt was presented by the Rev. Dr. Walsh, H. M., from Captain Walsh.

February 24, 1845.

SIR Wm. R. HAMILTON, LL.D., President, in the Chair.

Matthew Baker, Esq., Patrick Joseph Blake, Esq., John D'Arcy, Esq., Rev. Nicholas John Halpin, Samuel Haughton, Esq., F. T. C. D., William Hogan, Esq., C. E., James Macdonnell, Esq., C. E., Right Hon. David R. Pigot, Matthew R. Sausse, Esq., Walter Sweetman, Esq., R. William Town-